Math Refresher

A little basic mathematics is needed to appreciate much of physical science. This review is included mainly to help those readers whose mathematical skills have become rusty, but it is sufficiently self-contained to introduce such useful ideas as powers-of-10 notation to those who have not been exposed to them elsewhere.

Algebra

Algebra is the arithmetic of symbols that represent numbers. Instead of being limited to relationships among specific numbers, algebra can show more general relationships among quantities whose numerical values need not be known.

To give an example, in the theory of relativity it is shown that the "rest energy" of any object—that is, the energy it has due to its mass alone—is

$$E = mc^2$$

What this formula does is give a way to calculate the rest energy E in terms of mass m and speed of light c. The formula is not restricted to a particular object, but can be applied to any object whose mass is known. What is being shown is the way in which rest energy E varies with mass m *in general*.

If we are told only that the rest energy E of some object is 5 joules, we do not know upon what factors E depends or precisely how the value of E varies with those factors. (The joule is a unit of energy widely used in physics; it is equal to $1\,\text{kg} \cdot \text{m}^2/\text{s}^2$.) The quantities E and m are **variables,** since they have no fixed values. On the other hand, c^2 is a **constant,** since it is the square of the speed of light c and the value of c—almost 300 million m/s, or about 186,000 mi/s—is the same everywhere in the universe. Thus the formula $E = mc^2$ tells us, in a simple and straightforward way, that the rest energy of something varies only with its mass and also how to find the numerical value of E if we are given the mass m of a particular object.

The convenience of algebra in science is increased by the use of standard symbols for constants of nature. Thus c always represents the speed of light, π always represents the ratio between the circumference and diameter of a circle, e always represents the electric charge of the electron, and so on.

Before we go further, it is worth reviewing how the arithmetical operations of addition, subtraction, multiplication, and division are expressed in algebra. Addition and subtraction are straightforward:

$$x + y = a$$

means that we obtain the sum a by adding the two quantities x and y together, while

$$x - y = b$$

means that we obtain the difference b when quantity y is subtracted from quantity x.

In algebraic multiplication, no special sign is ordinarily used, and the symbols of the quantities to be multiplied are merely written together. Thus these four expressions have the same meaning:

$$xy = c \qquad x(y) = c \qquad (x)(y) = c \qquad x \times y = c$$

When the quantity x is to be divided by y to yield the quotient e, we write

$$\frac{x}{y} = e$$

which can also be expressed as

$$x/y = e$$

whose meaning is the same.

If several operations are to be performed in a certain order, parentheses ( ), brackets [ ], and braces { } are used to indicate this order. For instance $a(x + y)$ means that we are first to add x and y together and then to multiply their sum $(x + y)$ by a. In essence $a(x + y)$ is an abbreviation for the same quantity written out in full:

$$a(x + y) = ax + ay$$

Let us find the value of

$$v = 5\left[\frac{(x - y)}{z}\right] + w$$

when $x = 15$, $y = 3$, $z = 4$, and $w = 10$. We proceed as follows:

1. Subtract y from x to give

$$x - y = 15 - 3 = 12$$

2. Divide $(x - y)$ by z to give

$$\frac{(x - y)}{z} = \frac{12}{4} = 3$$

3. Multiply $[(x - y)/z]$ by 5 to give

$$5\left[\frac{(x - y)}{z}\right] = 5 \times 3 = 15$$

4. Add w to $5[(x - y)/z]$ to give

$$v = 5\left[\frac{(x - y)}{z}\right] + w = 15 + 10 = 25$$

Positive and Negative Quantities

The rules for multiplying and dividing positive and negative quantities are simple. If the quantities are both positive or both negative, the result is positive; if one is positive and the other negative, the result is negative.

In symbols,

$$(+a)(+b) = (-a)(-b) = +ab$$
$$\frac{+a}{+b} = \frac{-a}{-b} = +\frac{a}{b}$$
$$(-a)(+b) = (+a)(-b) = -ab$$
$$\frac{-a}{+b} = \frac{+a}{-b} = -\frac{a}{b}$$

Here are some examples:

$$(-3)(-5) = 15 \qquad \frac{-16}{-4} = 4$$
$$2(-4) = -8 \qquad \frac{10}{-5} = -2$$
$$(-12)6 = -72 \qquad \frac{-24}{4} = -6$$

To find the value of

$$w = \frac{xy}{x + y}$$

when $x = 5$ and $y = -6$, we begin by finding xy and $x + y$. These are

$$xy = (5)(-6) = -30$$
$$x + y = 5 + (-6) = 5 - 6 = -1$$

Hence

$$w = \frac{xy}{x + y} = \frac{-30}{-1} = 30$$

An example of the use of positive and negative quantities occurs in physics, where there are two kinds of electric charge. One kind is designated positive and the other negative. The force F that a charge Q_1 exerts on another charge Q_2 a distance r away is given by Coulomb's law as

$$F = K\frac{Q_1Q_2}{r^2}$$

where K is a universal constant. By convention, a positive value of F means a repulsion between the charges—the force tends to push Q_1 and Q_2 apart. A negative value of F means an attraction between the charges—the force tends to pull Q_1 and Q_2 together. A positive (=repulsive) force acts when *either* both charges are + *or* both are −: "like charges repel." When one charge is + and the other one −, the force is negative (=attractive): "opposite charges attract." Both the above observations about the types of force that occur together with the way in which the strength of F varies with the magnitudes of Q_1 and Q_2 and with their separation r are included in the simple formula $F = KQ_1Q_2/r^2$.

Exercises

A. Evaluate the following. The answers are given at the end of the Math Refresher.

1. $\dfrac{3(x + y)}{2}$ when $x = 5$ and $y = -2$

2. $\dfrac{1}{x - y} - \dfrac{1}{x + y}$ when $x = 3$ and $y = 2$

3. $\dfrac{4xy}{y + 3x} + 5$ when $x = 1$ and $y = -2$

4. $\dfrac{x + y}{2z} + \dfrac{z}{x - y}$ when $x = -2$, $y = 2$, and $z = 4$

5. $\dfrac{x + z}{y} + \dfrac{xy}{2}$ when $x = -2$, $y = 8$, and $z = 10$

6. $\dfrac{3(x + 7)}{y + 2}$ when $x = 3$ and $y = -6$

7. $\dfrac{5(3 - x)}{2(x + y)}$ when $x = -5$ and $y = 7$

Equations

An equation is a statement of equality: whatever is on the left-hand side of an equation is equal to whatever is on the right-hand side. An example of an arithmetical equation is

$$3 \times 9 + 8 = 35$$

since it contains only numbers. An example of an algebraic equation is

$$5x - 10 = 20$$

since it contains a symbol as well as numbers.

The symbols in an algebraic equation usually must have only certain values if the equality is to hold. To *solve* an equation is to find the possible values of these symbols. The solution of the equation $5x - 10 = 20$ is $x = 6$ since only when $x = 6$ is this equation a true statement:

$$\begin{aligned} 5x - 10 &= 20 \\ 5 \times 6 - 10 &= 20 \\ 30 - 10 &= 20 \\ 20 &= 20 \end{aligned}$$

The methods that can be used to solve an equation are based on this principle:

Any operation carried out on one side of an equation must be carried out on the other side as well.

Thus an equation remains valid when the same quantity is added to or subtracted from both sides or when the same quantity is used to multiply or divide both sides.

Two helpful rules follow from the above principle. The first is:

Any term on one side of an equation may be shifted to the other side by changing its sign.

To check this rule, let us consider the equation

$$a + b = c$$

If we subtract b from each side of the equation, we obtain

$$a + b - b = c - b$$
$$a = c - b$$

Thus b has disappeared from the left-hand side and $-b$ is now on the right-hand side. Similarly, if

$$a - d = e$$

then it is true that

$$a = e + d$$

The second rule is:

A quantity that multiplies one side of an equation may be shifted so as to divide the other side, and vice versa.

To check this rule, let us consider the equation

$$ab = c$$

If we divide both sides of the equation by b, we obtain

$$\frac{ab}{b} = \frac{c}{b}$$
$$a = \frac{c}{b}$$

Thus b, which was a multiplier on the left-hand side, is now a divisor on the right-hand side. Similarly, if

$$\frac{a}{d} = e$$

then it is true that

$$a = ed$$

Let us use the above rules to solve $5x - 10 = 20$ for the value of x. What we want is to have just x on the left-hand side of the equation. The first step is to shift the -10 to the right-hand side, where it becomes $+10$:

$$5x - 10 = 20$$
$$5x = 20 + 10 = 30$$

Now we shift the 5 so that it divides the right-hand side:

$$5x = 30$$
$$x = \frac{30}{5} = 6$$

The solution is $x = 6$.

When each side of an equation consists of a fraction, all we need do is **cross multiply** to remove the fractions:

$$\frac{a}{b} = \frac{c}{d} \qquad \frac{a}{b} \times \frac{c}{d} \qquad ad = bc$$

For practice, let us solve the equation

$$\frac{5}{a + 2} = \frac{3}{a - 2}$$

for the value of a. We proceed as follows:

Cross multiply to give	$5(a - 2) = 3(a + 2)$
Multiply out both sides to give	$5a - 10 = 3a + 6$
Shift the -10 and the $3a$ to give	$5a - 3a = 6 + 10$
Carry out the indicated addition and subtraction to give	$2a = 16$
Divide both sides by 2 to give	$a = 8$

Exercises

B. Solve each of the following equations for the value of x:

1. $3x + 7 = 13$

2. $5x - 8 = 17$

3. $2(x + 5) = 6$

4. $7x - 10 = 0.5$

5. $\frac{x + 7}{6} = x + 2$

6. $\frac{4x - 35}{3} = 9(1 - x)$

7. $\frac{3x - 42}{9} = 2(7 - x)$

8. $\frac{1}{x + 1} = \frac{1}{2x - 1}$

9. $\frac{3}{x - 1} = \frac{5}{x + 1}$

10. $\frac{1}{3x + 4} = \frac{2}{x + 8}$

Exponents

There is a convenient shorthand way to express a quantity that is to be multiplied by itself one or more times. In this scheme a superscript number called an **exponent** is used to show how many times the multiplication is to be carried out, as follows:

$$a = a^1$$
$$a \times a = a^2$$
$$a \times a \times a = a^3$$
$$a \times a \times a \times a = a^4$$

and so on. The quantity a^2 is read "a squared" because it is equal to the area of a square whose sides are a long. The quantity a^3 is read as

"a cubed" because it is equal to the volume of a cube whose edges are a long. Past an exponent of 3 we read a^n as "a to the nth power," so that a^5 is "a to the fifth power."

Suppose we have a quantity raised to some power, say a^n, that is to be multiplied by the same quantity raised to another power, say a^m. In this event the result is that quantity raised to a power equal to the sum of the original exponents:

$$a^n \times a^m = a^n a^m = a^{n+m}$$

To convince ourselves that this is true, we can work out $a^3 \times a^4$:

$$(a \times a \times a)(a \times a \times a \times a) = a \times a \times a \times a \times a \times a \times a$$
$$a^3 a^4 = a^7$$

Because the process of multiplication is basically one of repeated addition,

$$(a^n)^m = a^{nm}$$

where $(a^n)^m$ means that a^n is to be multiplied by itself the number of times indicated by the exponent m. Thus

$$(a^2)^4 = a^{2 \times 4} = a^8$$

because

$$(a^2)^4 = a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2} = a^8$$

Reciprocal quantities are expressed according to the above scheme but with negative exponents:

$$\frac{1}{a} = a^{-1} \qquad \frac{1}{a^2} = a^{-2} \qquad \frac{1}{a^3} = a^{-3} \qquad \frac{1}{a^4} = a^{-4}$$

Roots

When the **square root** of a quantity is multiplied by itself, the product is equal to the quantity. The usual symbol for the square root of a quantity a is $\sqrt{a}$. Thus

$$\sqrt{a} \times \sqrt{a} = a$$

Here are some examples of square roots:

$\sqrt{1} = 1$	because	$1 \times 1 = 1$
$\sqrt{4} = 2$	because	$2 \times 2 = 4$
$\sqrt{9} = 3$	because	$3 \times 3 = 9$
$\sqrt{100} = 10$	because	$10 \times 10 = 100$
$\sqrt{30.25} = 5.5$	because	$5.5 \times 5.5 = 30.25$
$\sqrt{16B^2} = 4B$	because	$4B \times 4B = 16B^2$

In the case of a number smaller than 1, the square root is larger than the number itself:

$\sqrt{0.01} = 0.1$	because	$0.1 \times 0.1 = 0.01$
$\sqrt{0.25} = 0.5$	because	$0.5 \times 0.5 = 0.25$

Similarly, multiplying the **cube root** $\sqrt[3]{a}$ of a quantity a by itself twice yields the quantity:

$$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$$

An expression of the form $\sqrt[n]{a}$ is read as "the nth root of a"; for instance, $\sqrt[4]{16}$ is "the fourth root of 16" and is equal to 2 because $2 \times 2 \times 2 \times 2 = 16$.

Although procedures exist for finding square and cube roots arithmetically, in practice electronic calculators or printed tables are normally used nowadays.

Here is an example of how a square root arises naturally in physics. Let us solve for r the equation

$$F = K\frac{Q_1Q_2}{r^2}$$

which expresses Coulomb's law of electric force. What we do is this:

Multiply both sides by r^2 to give $$Fr^2 = KQ_1Q_2$$

Divide both sides by F to give $$r^2 = \frac{KQ_1Q_2}{F}$$

Take square root of both sides to give $$r = \sqrt{\frac{KQ_1Q_2}{F}}$$

In algebra, a fractional exponent is used to indicate a root of a quantity. In terms of exponents we would write the square root of a as

$$\sqrt{a} = a^{1/2}$$

because

$$a^{1/2} \times a^{1/2} = (a^{1/2})^2 = a^{2\times 1/2} = a^1 = a$$

In a similar way the "cube root" of a, which is $\sqrt[3]{a}$, is indicated by the exponent $\frac{1}{3}$ because

$$a^{1/3} \times a^{1/3} \times a^{1/3} = (a^{1/3})^3 = a^1 = a$$

In general, the nth root of any quantity is indicated by the exponent $1/n$:

$$\sqrt[n]{a} = a^{1/n}$$

A few examples will indicate how fractional exponents fit into the general pattern of exponential notation:

$$(a^6)^{1/2} = a^{(1/2)\times 6} = a^3$$
$$(a^{1/2})^6 = a^{6\times 1/2} = a^3$$
$$(a^3)^{-1/3} = a^{(-1/3)\times 3} = a^{-1}$$
$$a^6a^{1/2} = a^{6+1/2} = a^{6\ 1/2}$$

Powers of 10

There is a convenient and widely used method for expressing very large and very small numbers that makes use of powers of 10. Any number in decimal form can be written as a number between 1 and 10 multiplied by some power of 10. The power of 10 is positive for numbers larger than 10 and negative for numbers smaller than 1. Positive powers of 10 follow this pattern:

$$
\begin{aligned}
10^0 &= 1 && = 1 \text{ with decimal point moved 0 places}\\
10^1 &= 10 && = 1 \text{ with decimal point moved 1 place to the right}\\
10^2 &= 100 && = 1 \text{ with decimal point moved 2 places to the right}\\
10^3 &= 1000 && = 1 \text{ with decimal point moved 3 places to the right}\\
10^4 &= 10{,}000 && = 1 \text{ with decimal point moved 4 places to the right}\\
10^5 &= 100{,}000 && = 1 \text{ with decimal point moved 5 places to the right}\\
10^6 &= 1{,}000{,}000 && = 1 \text{ with decimal point moved 6 places to the right}
\end{aligned}
$$

and so on. The exponent of 10 in each case indicates the number of places through which the decimal point is moved to the right from 1.00000 . . . . Equivalently, the exponent gives the number of zeroes that follow the 1.

Negative powers of 10 follows a similar pattern:

$$
\begin{aligned}
10^0 &= 1 && = 1 \text{ with decimal point moved 0 places}\\
10^{-1} &= 0.1 && = 1 \text{ with decimal point moved 1 place to the left}\\
10^{-2} &= 0.01 && = 1 \text{ with decimal point moved 2 places to the left}\\
10^{-3} &= 0.001 && = 1 \text{ with decimal point moved 3 places to the left}\\
10^{-4} &= 0.000{,}1 && = 1 \text{ with decimal point moved 4 places to the left}\\
10^{-5} &= 0.000{,}01 && = 1 \text{ with decimal point moved 5 places to the left}\\
10^{-6} &= 0.000{,}001 && = 1 \text{ with decimal point moved 6 places to the left}
\end{aligned}
$$

and so on. Here the exponent of 10 in each case shows the number of places through which the decimal point is moved to the left from 1. The number of zeroes between the decimal point and the 1 is one less than the exponent, that is, $n - 1$.

Here are some examples of powers-of-10 notation:

$$
\begin{aligned}
8000 &= 8 \times 1000 = 8 \times 10^3\\
347 &= 3.47 \times 100 = 3.47 \times 10^2\\
8{,}700{,}000 &= 8.7 \times 1{,}000{,}000 = 8.7 \times 10^6\\
0.22 &= 2.2 \times 0.1 = 2.2 \times 10^{-1}\\
0.000{,}035 &= 3.5 \times 0.000{,}01 = 3.5 \times 10^{-5}
\end{aligned}
$$

An advantage of powers-of-10 notation is that it makes calculations involving large and small numbers easier to carry out. The rules for working with exponents that were reviewed in the previous section hold for exponents of 10. We have here

Multiplication: $10^n \times 10^m = 10^{n+m}$

Division: $\dfrac{10^n}{10^m} = 10^{n-m}$

Raising to power: $(10^n)^m = 10^{nm}$

Taking a root: $(10^n)^{1/m} = 10^{n/m}$

An example will show how a calculation involving powers of 10 is worked out:

$$\begin{aligned}\frac{460 \times 0.000{,}03 \times 100{,}000}{9000 \times 0.006{,}2} &= \frac{(4.6 \times 10^2) \times (3 \times 10^{-5}) \times (10^5)}{(9 \times 10^3) \times (6.2 \times 10^{-3})} \\ &= \frac{4.6 \times 3}{9 \times 6.2} \times \frac{10^2 \times 10^{-5} \times 10^5}{10^3 \times 10^{-3}} \\ &= 0.25 \times \frac{10^{2-5+5}}{10^{3-3}} = 0.25 \times \frac{10^2}{10^0} \\ &= 25\end{aligned}$$

Another virtue of this notation is that it permits us to express the accuracy with which a quantity is known in a clear way. The speed of light in free space c is often given as simply 3×10^8 m/s. If c were written out as 300,000,000 m/s we might be tempted to think the speed is precisely equal to this number, right down to the last zero. Actually, the speed of light is 299,792,458 m/s. For our purposes we do not need this much detail. By writing just $c = 3 \times 10^8$ we automatically indicate both how large the number is (the 10^8 tells how many decimal places are present) and how precise the quoted figure is (the single digit 3 means that c is closer to 3×10^8 than it is to either 2×10^8 or 4×10^8m/s). If we wanted more precision, we could write $c = 2.998 \times 10^8$m/s. Again how large c is and how precise the quoted figure is are both obvious at a glance.

To be sure, sometimes one or more zeroes in a number are meaningful in their own right and not solely decimal-point indicators. In the case of the speed of light, we can legitimately state that, to three-digit accuracy

$$c = 3.00 \times 10^8 \text{m/s}$$

since c is closer to this figure than to 2.99×10^8 or 3.01×10^8 m/s. In the last sample calculation, the quantity $(4.6 \times 3)/(9 \times 6.2)$ actually equals 0.2473118. . . . It is rounded off to 0.25 because the result of a calculation may have no more significant digits than those in the least precise of the numbers that went into it.

Exercises

C. Express the following numbers in powers-of-10 notation:

1. 720 =	**2.** 890,000 =
3. 0.02 =	**4.** 0.000,062 =
5. 3.6 =	**6.** 0.4 =
7. 49,527 =	**8.** 0.002,943 =
9. 0.0014 =	**10.** 49,000,000,000 =
11. 0.000,000,011 =	**12.** 1.4763 =

D. Express the following numbers in decimal notation:

1. 3×10^{-4} =	**2.** 7.5×10^3 =
3. 8.126×10^{-5} =	**4.** 1.01×10^8 =

5. $5 \times 10^2 =$ **6.** $3.2 \times 10^{-2} =$

7. $4.32145 \times 10^3 =$ **8.** $6 \times 10^6 =$

9. $5.7 \times 10^0 =$ **10.** $6.9 \times 10^{-5} =$

E. Evaluate the following in powers-of-10 notation:

1. $\dfrac{30 \times 80{,}000{,}000{,}000}{0.0004} =$

2. $\dfrac{30{,}000 \times 0.000{,}000{,}6}{1000 \times 0.02} =$

3. $\dfrac{0.0001}{60{,}000 \times 200} =$

4. $5000 \times 0.005 =$

5. $\dfrac{5000}{0.005} =$

6. $\dfrac{200 \times 0.000{,}04}{400{,}000} =$

7. $\dfrac{0.002 \times 0.000{,}000{,}05}{0.000{,}004} =$

8. $\dfrac{500{,}000 \times 18{,}000}{9{,}000{,}000} =$

Answers

A.
1. 4.5
2. 0.8
3. −3
4. −1
5. 7
6. −7.5
7. 10

B.
1. 2
2. 5
3. −2
4. 1.5
5. −1
6. 2
7. 8
8. 2
9. 4
10. 0

C.
1. 7.2×10^2
2. 8.9×10^5
3. 2×10^{-2}
4. 6.2×10^{-5}
5. 3.6×10^0
6. 4×10^{-1}
7. 4.9527×10^4
8. 2.943×10^{-3}
9. 1.4×10^{-3}
10. 4.9×10^{10}
11. 1.1×10^{-8}
12. 1.4763×10^0

D.
1. 0.0003
2. 7500
3. 0.000,081,26
4. 101,000,000
5. 500
6. 0.032
7. 4321.45
8. 6,000,000
9. 5.7
10. 0.000,069

E.
1. 6×10^{15}
2. 9×10^{-4}
3. 8.3×10^{-11}
4. 2.5×10^1
5. 10^6
6. 2×10^{-8}
7. 2.5×10^{-5}
8. 10^3